FIRST PART: (Nash) Equilibria

(Some) Types of games

- Cooperative/Non-cooperative
- Symmetric/Asymmetric (for 2-player games)
- Zero sum/Non-zero sum
- Simultaneous/Sequential
- Perfect information/Imperfect information
- One-shot/Repeated

Games in Normal-Form

We start by considering simultaneous, perfect-information and non-cooperative games. These games are usually represented **explicitly** by listing all possible strategies and corresponding payoffs of all players (this is the so-called normal-form); more formally, we have:

- A set of N **rational** players
- For each player i, a **strategy set** S_i
- A **payoff matrix**: for each strategy combination $(s_1, s_2, \ldots, s_N)$, where $s_i \in S_i$, a corresponding payoff vector $(p_1, p_2, \ldots, p_N)$

$$\Rightarrow \quad |S_1| \times |S_2| \times \ldots \times |S_N| \text{ payoff matrix}$$

A famous game: the Prisoner's Dilemma

Non-cooperative, symmetric, non-zero sum, simultaneous, perfect information, one-shot, 2-player game

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

Strategy Set (Prisoner II: Don't Implicate, Implicate)

Strategy Set (Prisoner I: Don't Implicate, Implicate)

Payoffs (for this game, these are years in jail, so they should be seen as a cost that a player wants to minimize)

Prisoner I's decision

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

- Prisoner I's decision:
  - If II chooses Don't Implicate then it is best to Implicate
  - If II chooses Implicate then it is best to Implicate
  - It is best to Implicate for I, **regardless** of what II does: *Dominant Strategy*

Prisoner II's decision

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

- Prisoner II's decision:
  - If I chooses Don't Implicate then it is best to Implicate
  - If I chooses Implicate then it is best to Implicate
  - It is best to Implicate for II, **regardless** of what I does: *Dominant Strategy*

Hence…

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

- It is best for both to implicate **regardless** of what the other one does
- **Implicate** is a Dominant Strategy for both
- (Implicate, Implicate) becomes the **Dominant Strategy Equilibrium**
- Note: If they might collude, then it's beneficial for both to Not Implicate, but it's not an equilibrium as both have incentive to deviate

Dominant Strategy Equilibrium

- **Dominant Strategy Equilibrium:** is a strategy combination $s^* = (s_1^*, s_2^*, \ldots, s_i^*, \ldots, s_N^*)$, such that s_i^* is a **dominant strategy** for each i, namely, for any possible alternative strategy profile $s = (s_1, s_2, \ldots, s_i, \ldots, s_N)$:
  - if p_i is a utility, then $p_i(s_1, s_2, \ldots, s_i^*, \ldots, s_N) \geq p_i(s_1, s_2, \ldots, s_i, \ldots, s_N)$
  - if p_i is a cost, then $p_i(s_1, s_2, \ldots, s_i^*, \ldots, s_N) \leq p_i(s_1, s_2, \ldots, s_i, \ldots, s_N)$
- Dominant Strategy is the *best* response to *any* strategy of other players
- If a game has a DSE, then players will immediately converge to it
- Of course, **not all** games (only **very few** in the practice!) have a dominant strategy equilibrium

A more relaxed solution concept: Nash Equilibrium [1951]

- **Nash Equilibrium:** is a strategy combination $s^*= (s_1^*, s_2^*, \dots, s_N^*)$ such that for each i, s_i^* is a best response to $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_N^*)$, namely, for any possible alternative strategy s_i of player i
  - if p_i is a utility, then $p_i (s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*) \geq p_i (s_1^*, s_2^*, \dots, s_i, \dots, s_N^*)$
  - if p_i is a cost, then $p_i (s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*) \leq p_i (s_1^*, s_2^*, \dots, s_i, \dots, s_N^*)$

Nash Equilibrium

- In a NE no player can unilaterally deviate from his strategy given others' strategies as fixed
- Each player has to take into consideration the strategies of the other players
- If a game has one or more NE, players need not to converge to it
- Dominant Strategy Equilibrium $\Rightarrow$ Nash Equilibrium (but the converse is not true)

Nash Equilibrium: The Battle of the Sexes (coordination game)

		Woman	
		Stadium	Cinema
Man	Stadium	2, 1	0, 0
	Cinema	0, 0	1, 2

- (Stadium, Stadium) is a NE: Best responses to each other
- (Cinema, Cinema) is a NE: Best responses to each other

☹ but they are not Dominant Strategy Equilibria … are we really sure they will eventually go out together????

A crucial issue in game theory: the existence of a NE

- Unfortunately, for pure strategies games (as those seen so far, in which each player, for each possible situation of the game, selects his action deterministically), it is easy to see that we cannot have a general result of existence
- In other words, there may be no, one, or many NE, depending on the game

A conflictual game: Head or Tail

		Player II	
		Head	Tail
Player I	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- Player I (row) prefers to do what Player II does, while Player II prefer to do the **opposite** of what Player I does!

⇒ In any configuration, one of the players prefers to change his strategy, and so on and so forth…thus, there are no NE!

On the existence of a NE

- However, when a player can select his strategy randomly by using a probability distribution over his set of possible pure strategies (mixed strategy), then the following general result holds:
- **Theorem (Nash, 1951):** *Any game with a finite set of players and a finite set of strategies has a NE of mixed strategies (i.e., there exists a profile of probability distributions for the players such that the expected payoff of each player cannot be improved by changing unilaterally the selected probability distribution).*
- **Head or Tail game**: if each player sets p(Head)=p(Tail)=1/2, then the expected payoff of each player is 0, and this is a NE, since no player can improve on this by choosing unilaterally a different randomization!