

FIRST PART: (Nash) Equilibria





(Some) Types of games

- Cooperative/Non-cooperative
- Symmetric/Asymmetric (for 2-player games)
- Zero sum/Non-zero sum
- Simultaneous/Sequential
- Perfect information/Imperfect information
- One-shot/Repeated



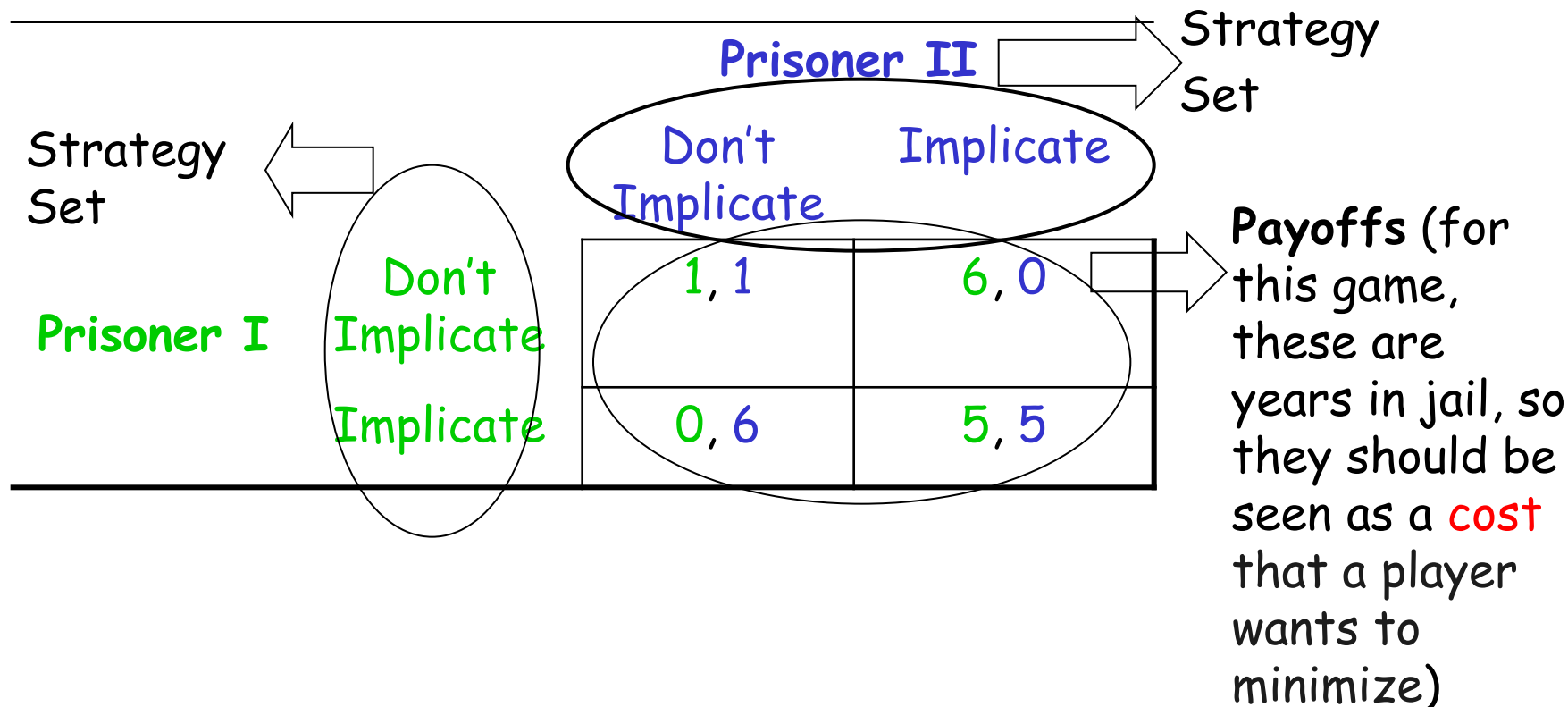
Games in Normal-Form

We start by considering **simultaneous, perfect-information** and **non-cooperative** games. These games are usually represented **explicitly** by listing all possible strategies and corresponding payoffs of all players (this is the so-called **normal-form**); more formally, we have:

- A set of N rational players
- For each player i , a **strategy set** S_i
- A **payoff matrix**: for each strategy combination (s_1, s_2, \dots, s_N) , where $s_i \in S_i$, a corresponding payoff vector (p_1, p_2, \dots, p_N)
 $\Rightarrow |S_1| \times |S_2| \times \dots \times |S_N|$ payoff matrix

A famous game: the Prisoner's Dilemma

Non-cooperative, symmetric, non-zero sum, simultaneous, perfect information, one-shot, 2-player game



Prisoner I's decision

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

- Prisoner I's decision:
 - If II chooses Don't Implicate then it is best to Implicate
 - If II chooses Implicate then it is best to Implicate
 - It is best to Implicate for I, regardless of what II does:
Dominant Strategy

Prisoner II's decision

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

- Prisoner II's decision:
 - If I chooses Don't Implicate then it is best to Implicate
 - If I chooses Implicate then it is best to Implicate
 - It is best to Implicate for II, regardless of what I does:
Dominant Strategy



Hence...

		Prisoner II	
		Don't Implicate	Implicate
Prisoner I	Don't Implicate	1, 1	6, 0
	Implicate	0, 6	5, 5

- It is best for both to implicate **regardless** of what the other one does
- **Implicate** is a Dominant Strategy for both
- (**Implicate**, **Implicate**) becomes the **Dominant Strategy Equilibrium**
- Note: If they might collude, then it's beneficial for both to **Not Implicate**, but it's not an equilibrium as both have incentive to deviate

Dominant Strategy Equilibrium

- **Dominant Strategy Equilibrium:** is a strategy combination $s^* = (s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*)$, such that s_i^* is a **dominant strategy** for each i , namely, for **any possible alternative strategy profile** $s = (s_1, s_2, \dots, s_i, \dots, s_N)$:
 - if p_i is a **utility**, then $p_i(s_1, s_2, \dots, s_i^*, \dots, s_N) \geq p_i(s_1, s_2, \dots, s_i, \dots, s_N)$
 - if p_i is a **cost**, then $p_i(s_1, s_2, \dots, s_i^*, \dots, s_N) \leq p_i(s_1, s_2, \dots, s_i, \dots, s_N)$
- Dominant Strategy is the *best response* to any strategy of other players
- If a game has a DSE, then players will immediately converge to it
- Of course, **not all games** (only **very few** in the practice!) have a dominant strategy equilibrium

A more relaxed solution concept: Nash Equilibrium [1951]

- **Nash Equilibrium:** is a strategy combination $s^* = (s_1^*, s_2^*, \dots, s_N^*)$ such that for each i , s_i^* is a best response to $(s_1^*, \dots, s_{i-1}^*, s_{i+1}^*, \dots, s_N^*)$, namely, for **any possible alternative strategy** s_i of player i
 - if p_i is a **utility**, then $p_i(s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*) \geq p_i(s_1^*, s_2^*, \dots, s_i, \dots, s_N^*)$
 - if p_i is a **cost**, then $p_i(s_1^*, s_2^*, \dots, s_i^*, \dots, s_N^*) \leq p_i(s_1^*, s_2^*, \dots, s_i, \dots, s_N^*)$



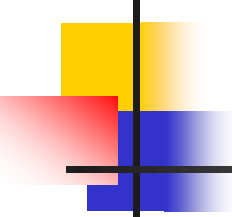
Nash Equilibrium

- In a NE no player can unilaterally deviate from his strategy given others' strategies as fixed
- Each player has to take into consideration the strategies of the other players
- If a game has one or more NE, players need not to converge to it
- Dominant Strategy Equilibrium \Rightarrow Nash Equilibrium (but the converse is not true)

Nash Equilibrium: The Battle of the Sexes (coordination game)

		Woman	
		Stadium	Cinema
Man	Stadium	2, 1	0, 0
	Cinema	0, 0	1, 2

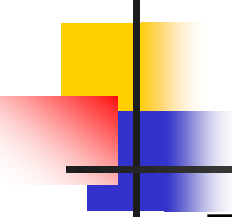
- (Stadium, Stadium) is a NE: Best responses to each other
- (Cinema, Cinema) is a NE: Best responses to each other
- ☹️ but they are not Dominant Strategy Equilibria ... are we really sure they will eventually go out together????



A crucial issue in game theory: the existence of a NE

- Unfortunately, for **pure strategies** games (as those seen so far, in which each player, for each possible situation of the game, selects his action **deterministically**), it is easy to see that we cannot have a general result of existence
- In other words, there may be **no**, **one**, or **many** NE, depending on the game

A conflictual game: Head or Tail



		Player II	
		Head	Tail
Player I	Head	1,-1	-1,1
	Tail	-1,1	1,-1

- **Player I** (row) prefers to do what **Player II** does, while **Player II** prefer to do the **opposite** of what **Player I** does!
- ⇒ In any configuration, one of the players prefers to change his strategy, and so on and so forth...thus, there are no NE!



On the existence of a NE

- However, when a player can select his strategy **randomly** by using a **probability distribution** over his set of possible pure strategies (**mixed strategy**), then the following general result holds:
- **Theorem (Nash, 1951)**: Any game with a finite set of players and a finite set of strategies has a NE of **mixed strategies** (i.e., there exists a profile of **probability distributions** for the players such that the **expected payoff** of each player cannot be improved by changing unilaterally the selected probability distribution).
- **Head or Tail game**: if each player sets $p(\text{Head})=p(\text{Tail})=1/2$, then the **expected payoff** of each player is 0, and this is a NE, since no player can improve on this by choosing unilaterally a different randomization!